BANGABASI COLLEGE

B.Sc. Third Year Honours Test Examination-2015

Subject – Mathematics

Paper - I

Full Marks – 100

Time – 4 Hours

1. Answer any one questions : 1x5=5

 (a) Let H be a subgroup of a cyclic group G. Prove that the quotient group G/H is cyclic. Is the converse true ? Justify your answer.

 (b) Let $\phi: (G,\circ)\rightarrow(G',*)$ be an isomorphism. Then prove that G' is commutative if and only if G is commutative. Also show that the groups $(Z_4,+)$ and V are not isomorphic.

 (c) Let H and K be subgroups of a group G. Let HK = {hk; h∈H, k ∈ K }. If K is normal in G ,then prove that HK is a subgroup of G and K is a normal subgroup of HK.

2. Answer any one questions : 1x5=5

 (a) Let V and W be vectors spaces over a field F and V is finite dimensional . If T:V→W be a linear mapping then prove that

 dimKerT + dim ImT = dimV.

 (b) The matrix of a linear mapping $T:\mathcal{R}^3 \rightarrow \mathcal{R}^3$relative to the ordered basis ((-1,1,1) ,(1,-1,1),(1,1,-1)) of $\mathcal{R}^3$ is$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find the matrix of T relative to the ordered basis ((0,1,1),(1,0,1),(1,1,0)) of $\mathcal{R}^3$.

 (c) Let T be a linear operator on V, where V is a vector space of dimension n over a field F. Prove that the following statements are equivalent:

 (i)T is nonsingular

 (ii)T maps a basis of V to another basis.

3. Answer any five questions : 5×5

 (a) If S⊂ $\mathcal{R}$ be such that every infinite subset of S has a limit point in S then prove that S is a compact set. Hence show that $\mathcal{R}$ is not a compact set.

 (b) If f : [a,b]→$\mathcal{R}$ has a derivative at every point x in [a,b] and f' is bounded on [a,b] then prove that f is a function of bounded variation. Is the converse true? Justify your answer.

(c) Let $f : [a,b] \to \mathcal{R}$ be a bounded function on [a,b]. Prove that anecessary and sufficient condition of integrabilityof f on [a,b] is that for every $\varepsilon>0$ there exists a partition P of [a,b] such that $U(P,f) - L(P,f) < \varepsilon$.

(d) Prove that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x+\cos x}\,dx < \frac{\pi^3}{24}$.

(e) For $x> -1$ and $x\neq 0$ prove that $\frac{x}{1+x} < \log(1+x) < x$. Hence show that

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

(f) Let X be a compact subset of $\mathcal{R}$and $f_n : X\to \mathcal{R}$ be a continuous function for each $n\in \mathcal{N}$. If $\{f_n\}$ converges pointwise to a continuous function f on X and the sequence $\{f_n\}$ is monotone for every $x \in X$ and for each $n \in \mathcal{N}$ then prove that $\{f_n\}$ converges uniformly to f on X.

(g) Determine the region of uniform convergence of the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n5^n}$.

4. Answer any two questions : 2x3.5

(a) If a_{ij} and b_k are the components of a symmetric covariant tensor and a covariant vector respectively which satisfy the relation $a_{jk}b_i+a_{ij}b_k+a_{ki}b_j = 0$,prove that either $a_{ij}= 0$ or $b_k= 0$.

(b) Find the Christoffel symbols of the second kind for the $\mathcal{V}_2$ with line element $ds^2= a^2(dx^1)^2+a^2\sin^2x^1(dx^2)^2$ where a is a constant.

(c) Prove that the covariant derivative of a tensor of type (1,0) is a tensor of type (1,1).

5. Answer any two questions: 2×4

(a) Use convolution theorem to show that

$L^{-1}\{\frac{1}{(p+2)^2(p-2)}\} = \frac{1}{16}(e^{2t} - 4te^{-2t} - e^{-2t})$.

(b) Solve the following differential equation using Laplace Transform

$(D^2 - D - 2)y = 20\sin 2t$, given $y=-1$, $Dy=2$ at $t=0$ where $D\equiv \frac{d}{dx}$.

(c) Solve the equation $(1-x^2)\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} - 4y = 0$ in series near the ordinary point x=0.

6. Answer any one question: 1x6=6

(a) Prove that $\vec{F} = (y^2\cos x + z^3)\hat{\imath} + (2y\sin x - 4)\hat{\jmath} + (3xz^2 + 2)\hat{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$. Find also the work done in moving a particle in this force field from (0,1,-1) to $(\frac{\pi}{2},-1,2)$.

(b) Verify Green's theorem in the plane for $\oint_C (x^2 - 2xy)dx + (x^2y + 3)dy$ around the boundary C of the region defined by $y^2 = 8x$ and $x = 2$.

(c) Using divergence theorem evaluate $\iint_S (3xz\hat{\imath} - 2y^2\hat{\jmath} + yz\hat{k}).\hat{n}ds$ where S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0,$ $and\ z = 1$ and $\hat{n}$ is the outward drawn unit normal to the surface.

7. Answer any two questions: 2x6=12

(a) If the moments and products of inertia oa a rigid body about three mutually perpendicular axes are known, find the moment of inertia of the body about any line through their meeting point.

(b) A bent lever whose arms are of length a and b, the angle between them being α, makes small oscillation in its own planeabout the fulcrum. Show that the length of the corresponding simple pendulum is $\frac{2}{3}.\frac{a^3+b^3}{\sqrt{a^4+2a^2b^2cos\alpha+b^4}}$

(c) A uniform rod of length 2a and weight W is turning about its end O and starts from the position in which it was vertically above the centre of inertia of the rod. When it was turned through an angle θ, show that the horizontal and vertical components of reactions of the axix of rotation on the rod at O are $\frac{3}{4}.Wsin\theta(2 - 3cos\theta)$ and $\frac{W}{4}.(1 - 3cos\theta)^2$.

(d) One end of a thread , which is wound in a reel is fixed and the reel falls in a vertical line , its axis being horizontal and the unwound part of the thread being vertical. If the reel b a solid cylinder of radius a and weight W, show that the acceleration of the centre of the reel is $\frac{2}{3}$g and the tension of the thread is $\frac{W}{3}$

8. Answer any two questions: 2x5 = 10

(a) A particle of mass m moves under a central attractive force $m\mu(5r^{-3} + 8c^2r^{-5})$ and is projected from an apse at a distance c with a velocity $\frac{3\sqrt{\mu}}{c}$. prove that the orbit is $r = ccos\frac{2}{3}\theta$. Show further that it will arrive at the origin after a time $\frac{\pi c^2}{8\sqrt{\mu}}$.

(b) If a planet was suddenly stopped in its orbit, supposed circlar, show that it will fall Into the Sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.

(c) A particle is projected horizontally with speed $\sqrt{\frac{ag}{2}}$ fro the highest point of the outside of a fixed smooth circle of radius a. show that it will leave the circle at the point whose vertical distance below the point of projection is $\frac{1}{6}a$.

(d) The volume of a spherical raindrop falling freely increases at each instant by an amount equal to μ times its surface area at that instant. If the initial radius of the drop be a , then show that its radius is doubled when it has fallen through a distance $\frac{9a^2g}{32\mu^2}$.

(e) Solve the following linear dynamical system:

$$\dot{x} = x + y, \quad \dot{y} = 4x - 2y$$

subject to the initial condition $(x_0,y_0) = (2,-3)$. Draw a phase portrait for the LDS and identify the critical point and discuss about its stability.

9. Answer any two questions: 2x5=10

(a) A square frame ABCD of four equal joined rods is hanging from A, the shape being maintained by a string joining mid points of AB and BC. Prove that the ratio of tension of the string to the reaction at C is $\frac{8}{\sqrt{5}}$.

(b) Establish the energy test of stability of equilibrium of a rigid body and explain it for one degree of freedom.

(c) Forces $\vec{P}$ *and* $\vec{Q}$ act along the straight lines : $y = xtan\alpha, z = c$ *and* $y = -xtan\alpha,\ z = -c$ respectively. Find the equation of their central axis and show that it is the generator of the surface $(x^2 + y^2)zsin2\alpha = 2cxy$.

10. Answer any two questions: 2x6=12

(a) Prove that if the forces per unit mass at (x,y,z) parallel to the axes are y(a-z), x(a-z) , xy, the surfaces of equal pressure are hyperbolic paraboloids and the curves of equal pressure and density are rectangular hyperbolas.

(b) A lamina in the shape of a quadrilateral ABCD has its side CD in the surface of a liquid and the sides AD, BC vertical and equal to α, β respectively. Show that the depth of its centre of pressure is $\frac{1}{2}\frac{(\alpha^2+\beta^2)(\alpha+\beta)}{\alpha^2+\alpha\beta+\beta^2}$

(c) A vessel in the form of a paraboloid of revolution formed by the revolution of a parabola of latus rectum 4a about its axis, is filled to half its height with liquid. Show that the greatest angular velocity with which it can revolve about its axis so that no liquid is spilt is $\frac{1}{4}\sqrt{\frac{6g}{a}}$.

(d) Show that a homogeneous right circular cone of vertical angle 2α cannot float stably with its axis vertical and vertex downwards unless its density as compared with that of the liquid is greater than $\cos^6\alpha$. What is the corresponding result when the vertex is upwards?

(e) Define Convective equilibrium in the atmosphear.
In such an atmosphere, show that the pressure p at any height z above the earth's surface is : $p = p_0[1 - \frac{\gamma-1}{\gamma}.\frac{1}{\rho_0{}^{\gamma-1}}.\frac{g}{k}z]^{\frac{\gamma}{\gamma-1}}$, where symbols used have their usual meanings and g has been assumed to be constant with height.

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BANGABASI COLLEGE

B.Sc. Third Year Honours Test Examination-2015

Subject – Mathematics

Paper - II

Full Marks – 75

Time – 4 Hours

1. Answer any two questions: 2x5=10

(a) Let X ⊂ IR and $\{f_n\}_n$ be a sequence of functions on X which converges pointwise to f on X. Let $M_n = \text{Sup}\, \{|f_n(x) - f(x)| : x \in X\}$. Prove that the sequence $\{f_n\}_n$ converges uniformly to f on X if and only if $M_n \to 0$ as $n \to \infty$.

(b) A sequence of functions $\{f_n\}_n$ is defined by $(x) = \sqrt{x^2 + \frac{1}{n^2}}$, $x \in [-1,1]$.

Show that $\{f_n\}_n$ is uniformly convergent on [-1,1]. Find the uniform limit f of the sequence $\{f_n\}_n$ and show that f is not differentiable on [-1,1], although f_n is differentiable for each n ∈ IN on [-1,1].

(c) Prove that the series $\sum_{n=1}^{\infty} \frac{sinnx}{n^4}$ is uniformly convergent for all real x. If s(x) be the sum function of the series, show that $s'(x) = \sum_{n=}^{\infty} \frac{\cos nx}{n^3}$, with proper justification.

(d) Determine the region of uniform convergence of the power series $\sum_{n=1}^{\infty} \frac{x^{n^n}}{n^n}$

2. Answer any five questions: 5x5=25

(a) State and prove Bayes' theorem.

(b) Find the most probable number of times the event 'multiple of three' occurs when a a die is thrown 100 times.

(c) The probability density function of a random variable X is Asech x. Find the value of A and compute P(X<1) and P(|X| ≥ 1).

(d) The joint density function of the random variables X and Y is given by ,
$f(x,y) = 2, \quad 0 < x < 1, \quad 0 < y < x.$
Find the marginal and conditional density functions.
Compute $P(\frac{1}{4} < X < \frac{3}{4} / Y = \frac{1}{2})$.

(e) If X and y are independent variates each uniformly distributed over the interval (-1,1), find the probability that the equation $t^2 + 2Xt + Y = 0$ has real roots.

(f) Show that th first absolute moment about the mean for the normal(m,σ) distribution Is $\sqrt{\frac{2}{\pi}}\sigma$.

(g) Define consistent and unbiased estimate of a population parameter. Show that sample variance is a consistent but not an unbiased estimate of population variance.

(h) Prove that the maximum likelihood estimate of the parameter α of a population having density function $2(\alpha - x)/\alpha^2$ $(0 < x < \infty)$ for a sample of unit size is 2x, x being the sample value , and show that the estimate is biased.

3. Answer any one question:

(a) (i) Show that the harmonic function u satisfies the differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$ 4

(ii) Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ 4

(b) (i) Show that a analytic function with constant modulus is constant. 4

(ii) Prove that $u = y^3 - 3x^2 y$ is a harmonic function.
Determine its harmonic conjugate and find the corresponding analytic function $f(z)$ in terms of z. 4

4. Answer any four questions: 8x4=32

[a] Let (X,d) be a complete metric space and $\{F_n\}$be any sequence of non-empty closed sets such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq$ .... in this space with $\underset{n\to\infty}{Lt}\ \delta(F_n) = 0$, where $\delta(A)$ denotes the diameter of the set A. Prove that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point in X.

[b] Let P[0,1] be the set of all polynomials defined on [0,1].

Show that $d(p_1, p_2) = \sup_{0 \le x \le 1} |p_1(x) - p_2(x)|$ is a metric on P[0,1]. Also show that this metric is incomplete.

[c] A committee of three persons decides proposals by a majority of votes. One member has a voting weight 1 and the others have weights 2,2 respectively. Design a simple circuit so that light will glow when and only when a majority of votes is cast in favour of a proposal.

[d] Write an efficient computer program in FORTRAN language to evaluate the integral $\int_1^2 \frac{e^x}{x} dx$ correct to 3 places of decimals using Simpson's one-third formula.

[e] Write an efficient computer program in FORTRAN language to solve numerically the differential equation $\frac{dy}{dx} = \frac{xy+1}{10y^2+1}$ for $x = 0(0.05)0.4$

by Runge-Kutta method correct upto 5 decimal places, given that $y = 0$ *for* $x = 0$.

[f] Deduce Lagrange's interpolation formula without error term.

[g] Explain the method of fixed point iteration to find a simple root of the equation f(x)=0. Derive the condition of convergence for the method.

[h] Describe the Power method to calculate numerically greatest eigen value of a real square matrix of order n.

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