BANGABASI COLLEGE

B.Sc. First Year (Part-I) Honours Test Examination-2015

Subject – Mathematics

Paper - I

Full Marks – 100

Time – 4 Hours

1. Answer any four questions : 4x5=20

(a) State Completeness property of IR.
Let S be a non-empty subset of IR with sup S =M and inf S = m. Prove that the set T = $\{|x-y| : x \in S, y \in S\}$ is bounded above and sup T = M - m.

(b) Define interior of a set S in IR. Prove that interior of S is the largest open set contained in S.

(c) Define a closed set. Prove that derived set of a set in IR is a closed set.

(d) Define an enumerable set. Prove that infinite subset of an enumerable set is enumerable.

(e) Define a monotone sequence. Prove that a monotone increasing sequence, if bounded above is convergent and it converges to its least upper bound.

(f) Prove that every bounded sequence of real numbers has a convergent subsequence.

(g) Let [a, b] be a closed and bounded interval and $f : [a, b] \to IR$ be continuous on [a, b]. If f(a) and f(b) are of opposite signs then prove that there exists at least one point c in the open interval (a, b) such that $f'(c) = 0$.

(h) Define Lipschitz function . Prove that a Lipschitz function $f : I \to IR$ is uniformly continuous on I.
If $f(x) = logx, x \in (0, \infty)$, show that f is uniformly continuous on $[a, \infty)$ where a>0.

2. Answer any one question: 1x5=5

(a) Evaluate the integral (Any one) :

(i) $\int \frac{xdx}{x^4-x^2-2}$; (ii) $\int_0^{\frac{\pi}{2}} \frac{xdx}{secx+cosecx}$

(b) If $I_n = \int_0^{\frac{\pi}{4}} tan^n \theta d\theta$, show that $I_n = \frac{1}{n-1} - I_{n-2}$. Hence find the value of $\int_0^{\frac{\pi}{4}} tan^6 \theta d\theta$.

3. Answer any two questions: 4x2=8

[a] Define Cartesian product of two non-empty sets.
Let A, B, C be three non-empty subsets of a set S.
Show that $A\times(B-C)=(A\times B)\bigcap(A\times C')$, where C' is the complement of C in S.

[b] Prove or disprove: $R=\{(a,b)\in Z\times Z : 3a+4b \text{ is divisible by } 7\}$ is an equivalence relation on Z, where Z is the set of integers.

[c] Let $f: A\to B$ and $g: B\to C$ be two mappings $(A, B, C$ *are non-empty sets*$)$.
Show that if $g\circ f$ is injective then f is injective, but g may not be so.

[d] Show that the set of complex numbers a+ib (where $i^2=-1$) for $a^2+b^2=1$ is a group under the multiplication of complex numbers.

4. Answer any two questions: 4x2=8

[a] A variable plane through the x-axis and a variable plane through the y-axis are inclined at a constant angle α. Prove that their lines of intersection generates the surface $z^2(x^2+y^2+z^2)=x^2y^2\tan^2\alpha$.

[b] A variable straight line, parallel to yz-plane intersects the curves $x^2+y^2=a^2, z=0$ and $x^2=az, y=0$. Prove that it generates the surface $x^4y^2=(a^3-x^2)(x^2-az)^2$.

[c] If θ be the angle between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2, show that the direction cosines of their angular bisectors are

$$\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}} \quad \text{and} \quad \frac{l_1-l_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}}.$$

[d] Show that the equation to the plane containing the straight line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to the straight line $\frac{x}{a}-\frac{z}{c}=1, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d be the shortest distance between the lines then show that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

5 . Answer any two questions: 4x2=8

[a] Show that a necessary and sufficient condition that a proper vector α always remains parallel to a fixed line is that $\alpha \times \frac{d\alpha}{dt}=0$.

[b] Prove that $div\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot curl\,\vec{A}-\vec{A}\cdot curl\,\vec{B}$.

[c] If F is a vector function, prove that $curl(curlF)=grad(divF)-LaplacianF$

[d] Prove , $\nabla^2 f(r)=\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}$, where $r=\sqrt{x^2+y^2+z^2}$. Find $f(r)$ such that $\nabla^2 f(r)=0$.

6. Answer any four questions: 4× 4

(a) If a and b are integers , not both zero , then prove that there exist integers u and v such that gcd(a,b) = au + bv.

(b) Find the least positive integer which leaves remainders 2 , 3 and 4 when divided by 3 , 5 and 11 respectively.

(c) The equation $3x^3+5x^2+5x+3=0$ has three distinct roots of equal moduli . Solve it.

(d) Find the relation among the coefficients of the equation $x^4+px^3+qx^2+rx+s=0$ if its roots $\alpha, \beta, \gamma, \delta$ be connected by the relation $\alpha\beta+1=0$.

(e) Use Strum's functions to show that the roots of the equation $x^4 + 4x^3 - x^2 - 10x + 3 = 0$ are all real and distinct.

(f) If α, β, γ be the roots of the equation $x^3 + 3x + 2 = 0$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\gamma}{\alpha}, \frac{\alpha}{\gamma}$.

(g) Show that the sum of the squares of all the values of $(\sqrt{3} + i)^{3/7}$ is zero.

(h) If a, b, c be all positive real numbers, then prove that $\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a + b + c$

7. Answer any two questions: 2× 5

(a) A point moves so that the distance between the feet of the perpendiculars from it on the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ is a constant 2d. Show that its locus is $(x^2 + y^2)(h^2 - ab) = d^2\{(a-b)^2 + 4h^2\}$.

(b) If A and B be two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that three times the eccentric angle of the one is equal to the supplement of that of the other, then find the locus of the pole of AB with respect to the ellipse.

(c) Show that the locus of the point of intersection of the tangents to the parabola $y^2 = 4ax$ at points whose ordinates are in the ratio $p^2 : q^2$ is the parabola
$y^2 = (\frac{p^2}{q^2} + \frac{q^2}{p^2} + 2)ax$.

(d) If the normal be drawn at one extremity $(l, \frac{\pi}{2})$ of the latus rectum LSL′ of the conic $\frac{l}{r} = 1 + e\cos\theta$ where S is the pole, then show that the distance from the focus S of the other point in which the normal meets the conic is $\frac{l(1+3e^2+e^4)}{1+e^2-e^4}$.

8. Answer all the questions:

(a) Prove that the medians of a triangle are concurrent. 4

(b) Prove by vector method that, in any triangle ABC

(i) $CosA = \frac{b^2+c^2-a^2}{2bc}$ and (ii) $c = aCosB + bCosA$. 5

9. Answer any four questions: 4x4=16

(a) Prove that $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.

(b) If A is a real skew-symmetric matrix and $(I + A)$ is non-singular, prove that the matrix $(I + A)^{-1}(I - A)$ is orthogonal.

(c) Determine the conditions for which the system of equations

$$x + y + z = b$$
$$2x + y + 3z = b + 1$$
$$5x + 2y + az = b^2$$

has (i) one solution; (ii) no solution; (iii) many solutions.

(d) Find the dimension of the subspace S of IR^4 defined by $S = \{(x, y, z, w) \in IR^4 : x + 2y - z = 0, 2x + y + w = 0\}$.

(e) (i) Prove that if λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

(ii) Find the eigen values and the corresponding eigen vectors of the matrix $\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$.

(f) Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ to the normal form and show that it is positive definite.

(g) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space IR^4 with standard inner product, generated by the linearly independent set $\{(1,1,0,1), (1,1,0,0), (0,1,0,1)\}$.

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