REAL ANALYSIS I

I Real number system

1 State Peano's axioms for the set of natural number. (2)

2. ~~Prove th~~ State wellordering principle in the set of natural numbers. Show that wellordering principle implies principle of mathematical induction (4)

3. Show that there does not exist a rational number r such that $r^2 = 5$. (3)

4. State least upper bound axiom in the set of real numbers Use it to prove that every non-empty set of real numbers which is bounded below has the infimum in the set of real numbers. (3)

5 Prove that the set N of natural numbers is not bounded above (3)

6 Let S be a nonempty subset of R, bounded below and $T = \{-x ; x \in S\}$ Prove that the set T is bounded above and $\sup T = -\inf S$ (3)

7 Let S and T be two nonempty bounded subsets of R. Let $U = \{x+y ; x \in S, y \in T\}$ Prove that $\inf U = \inf S + \inf T$. (3)

8. Let S be a bounded subset of R with $\sup S = M$, $\inf S = m$. Prove that the set $T = \{x-y ; x, y \in S\}$ is a bounded set and $\sup T = M - m$, $\inf T = m - M$

9. Show that the subset $S = \{x \in Q ; x > 0$ and $x^2 > 2\}$ is a nonempty subset of Q, bounded below, but $\inf S$ does not belong to Q. (4)

10. State and prove archimedian property for the set of real numbers. (3)

11. Prove that every open interval (a, b) $[a<b]$ contains an irrational number. (3)

12. Prove that for any real number $x > 0$, there exists a (+)ve integer m such that $m-1 \leq x < m$ (3)

II Sets in R

1. Define open set. Prove that union of two open sets is an open set (3)

2. Prove that the intersection of a finite number of open sets in R is an open set Is the result true for arbitrary collection of open sets? Justify your answer. (3)

3. Define interior of a set Prove that it is the largest open set contained in S (4)

4. Define derived set of a set. Prove that it is a closed set (4)

5. Find the derived set of the set
$$S = \{ \tfrac{1}{m} + \tfrac{1}{n} ; \ m, n \in N \}.$$ (3)

6. Let $S = \{ (-1)^m + \frac{1}{n} ; \ n \in N \}$. Show that -1 and 1 are limit points of S. (3)

7. Prove that ~~intersection~~ union of finite number of closed sets is a closed set Is the result true for arbitrary collection of closed set? Justify your answer (4)

8. Prove that every bounded infinite set of real numbers has at least one limit point (4)

9. State Bolzano Weierstrass theorem for the set of real numbers. Verify it for the set
$$S = \{ 1 + \tfrac{1}{n} ; \ n \in N \}$$ (3)

10. Let S be a nonempty subset of R bounded below and $m = \inf S$ Prove that either $m \in S$ or m is a limit point of S (3)

11. Let G be an open set in R and S be a subset of R such that $G \cap S = \varphi$. Prove that $G \cap S' = \varphi$. (3)

12. Let $G \subset R$ be an open set and $F \subset R$ be a closed set Prove that $G - F$ is an open set and $F - G$ is a closed set (3)

13. Prove that every finite set of real numbers is a closed set

14. Let $S = (-1, 1) \cup \{\pm 1, \pm 2, \pm 3, \quad \}$. Find the isolated points of S (3)

15. Let $f \; S \to R$ be a continuous function. Let $A = \{ x \in S ; f(x) > 0\}$ Prove that A is an open set. (3)

16. Give an example of an infinite set $S \subset R$ such that (i) S has no limit point

(ii) S has only one limit point. (3)

17 Prove that the set R is not enumerable. (3)

18. Prove that the set of all open intervals with rational end points is enumerable (3)

18. Let S be a subset of R such that no point of S is a limit point of S Prove that S is a countable set (3)

19 Prove that the set of positive rational numbers is enumerable (3)

20 Prove that the union of an enumerable number of enumerable sets is enumerable (3)

III Sequence

1. Define convergent sequence. Prove that a convergent sequence is bounded. Is the converse true? Justify your answer.

2. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that (4)

$\lim_{n\to\infty} u_n = u$ $\lim_{n\to\infty} v_n = v$ Prove that

$\lim_{n\to\infty} (u_n v_n) = uv$

(4)

3. State sandwich theorem for real sequences. Use it to prove that $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$

4. If $\lim_{n\to\infty} x_n = 0$, prove that $\lim_{n\to\infty} \log(1+x_n) = 0$ (3)

5. A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1}+u_n)$ for $n \geq 1$ and $0 < u_1 < u_2$ Prove that the sequence $\{u_n\}$ converges to $\frac{u_1+2u_2}{3}$ (3)

(3)

6. Define monotone increasing sequence. Prove that a monotone increasing sequence, if bounded above, is convergent and converges to the least upper bound.

(4)

7. State and prove Cantor's theorem on nested intervals. Is the result true for open intervals? Justify your answer

(5)

8. Prove that the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is monotone decreasing and is bounded below.

(4)

9. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6+u_n}$ for $n>1$ converges to 3.

(4)

10. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = \sqrt{x_n y_n}$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1$, $x_1 > 0$, $y_1 > 0$ converges to a common limit

(4)

11. Let S be a nonempty subset of R having a limit point ℓ. Show that there exists a sequence $\{u_n\}$ of distinct elements of S such that $\lim_{n\to\infty} u_n = \ell$.

(4)

12. Use sandwich theorem to prove that $\lim_{n\to\infty} (2^n+3^n)^{\frac{1}{n}} = 3$

(2)

13. Prove that the sequence $\{u_n\}$ is a null sequence where $u_n = \dfrac{\lfloor n}{n^n}$ $\forall n \in N$ (2)

14. Define Cauchy sequence. Prove that the sequence $\{u_n\}$ where $u_1 = 0$, $u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \geq 1$ is a Cauchy sequence

(4)

15. If $\underset{n\to\infty}{Lt}\, u_n = l$, prove that $\underset{n\to\infty}{Lt}\, \dfrac{u_1 + u_2 + \cdots + u_n}{n} = l$.

Hence show that $\underset{n\to\infty}{Lt}\, \dfrac{1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}}{n} = 1$

(4)

16. Prove that $\underset{n\to\infty}{Lt}\, \dfrac{(\lfloor n)^{\frac{1}{n}}}{n} = \dfrac{1}{e}$

(2)

17. Let $\{u_n\}$ be convergent sequence and $\underset{n\to\infty}{Lt}\, u_n = 0$ and let $\{v_n\}$ be a bounded sequence. Is the sequence $\{u_n v_n\}$ convergent? Justify your answer.

(3)

18. Prove that the sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not a Cauchy sequence

(3)

19. Prove that every bounded sequence of real numbers has a convergent subsequence.

(4)

20. If the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converges to the same limit l then the sequence $\{u_n\}$ is convergent and $\underset{n\to\infty}{Lt}\, u_n = l$

Hence show that the sequence $\{u_n\}$, defined by $u_n > 0$ and $u_{n+1} = \dfrac{6}{1+u_n}$ $\forall n \in N$ is convergent

(5)

21. Define upper limit and lower limit of a bounded real sequence. Find the upper and lower limit of the sequence $\{\sin \frac{n\pi}{2}\}$.

(4)

22. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences. Prove that $\underline{\lim}\, u_n + \underline{\lim}\, v_n \leq \underline{\lim}\,(u_n + v_n)$.

(3)

IV Series

1. Let Σu_n be a convergent series of positive real numbers Then show that any rearrangement of Σu_n is convergent and the sum remains same. (4)

2. Test the convergence of the series
$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots$$
(2)

3. Let Σu_n be a series of positive real numbers and let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ Show that Σu_n is convergent if $l<1$ and is divergent if $l>1$ What happens if $l=1$? Justify your answer (5)

4. Test the convergence of the series
$$1 + \frac{1}{\lfloor 1} + \frac{2^2}{\lfloor 2} + \frac{3^3}{\lfloor 3} + \cdots$$
(2)

5. Test the convergence of the series
$$a + b + a^2 + b^2 + a^3 + b^3 + \cdots \quad \text{where } 0<a<b<1$$
(3)

6. Let $\{f(n)\}$ be a monotone decreasing sequence of positive real numbers and a be a positive integer >1 Show that the series $\sum_1^\infty f(n)$ and $\sum_1^\infty a^n f(a^n)$ converge or diverge together. (4)

7. Test the convergence of the series
$$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7} + \cdots$$
(3)

8. Test the convergence of the series
$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \cdots$$
where $\alpha, \beta, \gamma, x > 0$ (4)

9. Let Σu_n be a convergent series of positive real numbers Prove that $\Sigma \frac{u_n}{1+u_n}$ is convergent (3)

10. Prove that an absolutely convergent series is convergent (3)

11. State and prove Leibnitz's test for alternating series (4)

12. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ Converges

to $\log 2$, but the rearranged series

$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$ converges to $\frac{1}{2}\log 2$.

13. Show that the series (4)

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots \qquad a>0$$

is (i) absolutely convergent if $p>1$

(ii) conditionally convergent if $0 < p \le 1$.

(4)

14 If $\{u_n\}$ be a sequence of real numbers and $\sum u_n^2$ is convergent prove that $\sum \frac{u_n}{n}$ is absolutely convergent

(3)

V Limit, continuity of a function of single variable

1. Let $f: D \to R$ be a function $(D \subset R)$ and $c \in D'$. Then Prove that $\underset{x \to c}{Lt} f(x) = l$ if and only if for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l (4)

2. Show that $\underset{x \to 0}{Lt} [x]$ does not exist (2)

3. ~~Let~~ State Cauchy's principle for existence of finite limit. Let $f: (0,1) \to R$ be defined by

$$f(x) = 1 \quad \text{if } x \in (0,1) \cap Q$$
$$= -1 \quad \text{if } x \in (0,1) - Q$$

Use Cauchy's principle to prove that $\underset{x \to a}{Lt} f(x)$ does not exist, $a \in [0,1]$ (4)

4. Prove that $\underset{x \to 0}{Lt} \frac{\sin x}{x} = 1$ (3)

5. Give an example of a function f defined over an interval I, in each case such that (i) f has jump discontinuity at a point of I (ii) f has removable discontinuity at a point of I (iii) f has infinite discontinuity at a point of I. (3)

6. Let f be defined in the interval I and c is an interior point of I. Let f be continuous at c. If $f(x)$ maintains both positive and negative signs in every neighbourhood of c, then state with reasons whether $f(c) = 0$ or $f(c) > 0$ or $f(c) < 0$ (3)

7. Let $f(x) = 1$ when x is rational
$= 0$ when x is irrational

Show that f is discontinuous every where. (3)

8. If f is a monotone increasing function defined on $[a,b]$ and $a < c < b$, show that both $f(c+0)$ and $f(c-0)$ exist finitely. What happens if $f(c+0) = f(c-0)$? (5)

9. Let f be a continuous function in $[0,1]$ and f assumes irrational values only. If $f(\frac{1}{2}) = \sqrt{2}$, show that $f(x) = \sqrt{2} \quad \forall x \in [0,1]$ (3)

10. If a real valued function f defined over $[a,b]$ is continuous therein and if $f(a)f(b)<0$, show that there exists at least one point c in (a,b) such that $f(c)=0$.

(5)

11. If f be a real valued continuous function on closed interval $[a,b]$, show that f is bounded on $[a,b]$. Is the result true for open interval ? Justify your answer

(4)

12. Given a nonempty set A of real numbers, define $d(x,A) = \inf\{|x-a| ; a \in A\}$ for every real numbers x Show that $g(x) = d(x,A)$ $\forall x \in R$ is continuous everywhere.

(3)

13. If $f : [a,b] \to R$ be continuous on $[a,b]$ and f is injective on $[a,b]$, show that f is strictly monotone on $[a,b]$

(4)

14 Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous in $(0,1)$, but is uniformly continuous in $(a,1)$ ~~if~~ where $0<a<1$

(3)

15 Let $f : D \to R$ $(D \subset R)$ be uniformly continuous on D. If $\{x_n\}$ be a cauchy sequence in D, show that $\{f(x_n)\}$ is a cauchy sequence

(3)

16 If $f(x+y) = f(x)+f(y)$ $\forall x,y \in R$ and f is continuous at a point of R, prove that f is uniformly continuous on R.

(3)

17 Prove that $\cos x = x^2$ for some $x \in (0, \frac{\pi}{2})$.

(3)

18. Give an example of a function which satisfies intermediate value property on $[a,b]$, but is not continuous on $[a,b]$.

(3)

VI Differentiation and related topics

1. If f is defined on the closed interval $[a,b]$ has a finite derivative at each point of (a,b), $f'(a+0)$ and $f'(b-0)$ exist and if $f'(a+0)f'(b-0)<0$, show that there exists at least one point c in (a,b) such that $f'(c)=0$. (5)

2. State and prove Lagrange's Mean value theorem. (4)

3. Let f be monotone increasing and differentiable function on $[a,b]$ and g be differentiable function on $[f(a),f(b)]$. If $\varphi(x) = g[f(x)]$ $\forall x \in [a,b]$, prove that
$\varphi'(x) = g'[f(x)]f'(x)$ $\forall x \in (a,b)$ (4)

4. If $f'(x)>0$ for all x in an interval I, prove that f is strictly increasing on I. (3)

5. State Rolle's theorem Examine the applicability of Rolle's theorem in $[-1,1]$ of the following functions
(i) $f(x) = |x|$ (ii) $g(x) = x^3$ (3)

6. If f is derivable in the open interval (a,b) and if f is continuous at the end points a, b and $f(a)=f(b)$, then show that $f'(x)=0$ has at least one root in (a,b) (3)

7. If f' exists on $[0,1]$, then show by Cauchy's Mean Value theorem that $f(1)-f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0,1)$ (3)

8. If f is differentiable in (a,b) and at a point $c\in(a,b)$ $\lim_{x\to c} f'(x) = l$ exists, then show that f' is continuous at c. (5)

9 Let $a_0+\frac{a_1}{2}+\frac{a_2}{3}+ \quad + \frac{a_n}{n+1}=0$ where a_i's are constants Show that the equation $a_0+a_1x+- \quad +a_nx^n=0$ has at least one solution in $(0,1)$ (3)

10. If $f'(x)$ exists and bounded on some interval I, prove that f is uniformly continuous on I. (3)

11. Let a function $f: R\to R$ satisfies the condition
$|f(x)-f(y)| \le (x-y)^2$ $\forall x,y \in R$
Prove that f is a constant function on R. (3)

12. If $f''(x)$ exists in $[a-h, a+h]$ prove that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(c) \text{ for some } c \in (a-h, a+h).$$ (3)

13. State and prove Maclaurin's theorem with Lagranges form of remainder. (5)

14. If f'' is continuous at a and $f''(a) \neq 0$ prove that $\underset{h \to 0}{Lt}\ \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)$ $0<\theta<1$. (3)

15. Obtain the Maclaurin's series expansion of $\log(1+x)$, $-1 < x \leq 1$ (4)

16. If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maximum nor a minimum at a and f has a minimum at b. (4)

17. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis and the minor axis at P, Q respectively. Show that the least value of PQ is $a+b$ (4)

18. Evaluate $\underset{x \to 0+}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ (3)

19. Find the value of a such that $\underset{x \to 0}{Lt} \frac{a \sin x - \sin 2x}{\tan^3 x}$ is finite Find the limit (3)

20. Find a and b such that $\underset{x \to 0}{Lt} \frac{ae^x + be^{-x} + 2\sin x}{\sin x + x\cos x} = 2$. (3)

21. If $f(x) = \frac{x^3}{x^2-1}$, prove that for $n>1$,

$f^n(0) = 0$ if n be even

$= -\lfloor n$ if n be odd (3)

22. Use Taylor's theorem to prove that

$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$. (3)

REAL ANALYSIS II

I Compact set

1. If $E \subset R$ be a compact set, then prove that E is closed and bounded (5)

2. Let $K \subset R$ be such that every open covering of K contains a finite subcovering Show that every infinite subset of K has a limit point in K Hence prove that R is not compact (3+1)

3. If A and B are respectively closed and compact subsets of the set of real numbers then show that $A \cap B$ is compact (3)

4. If K be a nonempty compact set in R, show that K has a least and greatest element. (4)

5. If $S \subset R$ be a compact set and $f\ S \to R$ be continuous on S, prove that f is uniformly continuous on S. (5)

6. Let $H = [0, \infty]$ and $G = \{G_n; n \in N\}$ where $G_n = (-1, n)$ Show that G is an open cover of H but no finite sub collection of G can cover H. Explain why H fails to satisfy Heine Borel property (3+1)

7. Construct a set $F \subset R$ such that every infinite subset of F has no limit point lying in F. (2)

8. Is the set of irrational numbers compact? Give reasons in support of your answer. (3)

9. Give an example of a set of real numbers which is bounded but not compact (2)

10. If a function f is continuous on a compact set $E \subset R$ and has an inverse f^{-1}, show that f^{-1} is continuous on $f(E)$. (3)

11. Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$, but uniformly continuous on $(a,1)$ where $0<a<1$ (4)

12. Prove that union of finite no. of compact sets in R is a compact set Is the result true for an arbitrary collection of compact sets in R? Justify your answer. (3+1)

13. For each $x \in (0,2)$, Let $I_x = \left(\frac{x}{2}, \frac{x+2}{2}\right)$. Show that the family $G = \{I_x ;\ x \in (0,2)\}$ is an open cover of the set $(0,2)$ Show that no finite subcollection of G can cover $(0,2)$

(4)

14 Prove that colosed subset of a compact set in R is compact

(3)

15. Let S be a closed and bounded subset of R Prove that every open cover of S has a finite subcover.

(5)

II Sequence and series of functions defined on a set.

1. If a sequence of functions $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and if $c \in [a,b]$ such that $\lim_{x\to c} f_n(x) = a_n$ $(n=1,2\ldots)$ show that
(i) the sequence $\{a_n\}$ converges
(ii) $\lim_{x\to c} f(x)$ exists (5)

2. Show that the sequence $\{f_n\}$ where
$$f_n(x) = \begin{cases} n^2x & 0 \le x \le \frac{1}{n} \\ -n^2x + 2n & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$
is not uniformly convergent on $[0,1]$ (3)

3. Let the sequence of functions $\{f_n\}$ converges uniformly to f on $[a,b]$ and each f_n is continuous on $[a,b]$. Prove that f is continuous on $[a,b]$ (4)

4 Let g $[0,1] \to R$ be a continuous function with $g(1)=0$. Show that the sequence of functions $\{f_n\}$ defined by $f_n(x) = x^n g(x)$ $(n=1,2\ldots)$ converges uniformly to a constant function zero. (3)

5. Let D be a finite subset of R If a sequence $\{f_n\}$ of real valued functions on D converges pointwise to f, prove that $\{f_n\}$ converges uniformly to f on D.

6. Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$. Show that $\{f_n\}$ converges pointwise to zero for all $x \in R$ but on any interval containing '0' $\{f_n\}$ is not uniformly convergent. (3) (4)

7. Prove that a sequence of function $\{f_n\}$ on a set E of real number converges uniformly on E to zero if and only if
$$\lim_{x\to\infty} \text{l.u.b}\{\,|f_n(x)|\,;\, x\in E\} = 0.$$
(4)

8. Let the sequence $\{f_n\}$ converge uniformly to f on the closed interval $[a,b]$ and let each f_n be Riemann integrable on $[a,b]$ Prove that f is Riemann integrable over $[a,b]$

9. ~~If~~ Let for each $n \in N$, f_n' is continuous on $[a,b]$. If $\{f_n\}$ converges in $[a,b]$ to f and if $\{f_n'\}$ converges uniformly to g on $[a,b]$, then show that $g(x) = f'(x)$, $x \in (a,b)$ (5)
(5)

10. Show that the sum of any uniformly convergent series of continuous functions is continuous (5)

11. State and prove Weierstrass M-test for uniform and absolute convergence of a series of functions

(5)

12. Use Weierstrass M-test to show that if a is a real number different from 0, ±1, ±2, ... then the series

$$\frac{1}{a} - \frac{\cos x}{a^2-1^2} + \frac{\cos 2x}{a^2-2^2} - \frac{\cos 3x}{a^2-3^2} + \cdots$$

converges uniformly in every finite interval of real line

(4)

13 Let $a<c<b$ Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,c]$ and $[c,b]$ Show that $\sum_{n=1}^{\infty} f_n(x)$ Converges uniformly on $[a,b]$

(4)

14. If the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent in the interval $[a,b]$ and each of the functions $f_n(x)$ is continuous in $[a,b]$, show that

$$\int_{c_1}^{c_2} \left\{ \sum_{n=1}^{\infty} f_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_{c_1}^{c_2} f_n(x)\, dx, \quad a \leq c_1 < c_2 \leq b.$$

(5)

15. Show by an example that uniform convergence is not a necessary condition for the continuity of the sum function of a series of functions

16 Show that the series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is not uniformly convergent in $(-\infty, \infty)$ but for any a with $0<a<\infty$ show that it is uniformly convergent in $[-c,c]$ (3)

17. If a sequence of continuous functions $\{f_n\}$ defined on $[a,b]$ is monotone increasing and converges pointwise to a continuous function f on $[a,b]$ Then show that the convergence is uniform in $[a,b]$ (4)

18. A function f is defined for $x>0$ as

$$f(x) = e^{-x} + 2e^{-2x} + \cdots + ne^{-nx} + \cdots$$ (5)

Show that

(i) f is continuous for $x>0$

(ii) the series can be integrated term by term in any interval in $(0,\infty)$

(iii) $\int_{\ln(2)}^{\ln(3)} f(x)\, dx = \frac{1}{2}$ (5)

III Power series

1 If a power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0$, then show that it is absolutely convergent for every $x = x_1$ where $|x_1| < |x_0|$. Also if it diverges for $x = x_0$, then show that it diverges for every $x = x_1$ where $|x_1| > |x_0|$ (4)

2. If a power series $\sum_{n=1}^{\infty} a_n x^n$ converges at the end point $x = R$ of the interval of convergence $(-R, R)$, then show that it is uniformly convergent in the closed interval $[0, R]$ (5)

3. Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$ $(-1 < x < 1)$ where $\sum_{n=0}^{\infty} c_n < \infty$. Show that

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$$

(3)

4. Let $\{a_n\}$ be a sequence of real numbers Prove that

(i) If $\limsup_{n \to \infty} \sqrt[n]{|a_n|} = 0$, then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for every real number x

(ii) If $\limsup_{n \to \infty} \sqrt[n]{|a_n|} = L > 0$, then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for $|x| < \frac{1}{L}$ but is not convergent for $|x| > \frac{1}{L}$ (5)

5. Find the radius of convergence of the following power series

$$\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots$$

(2)

6. Find the interval of convergence of the power series

$$x + \frac{x^2}{20} + \cdots + \frac{x^n}{n\,10^{n-1}} + \cdots$$

(3)

7. Define a power series indicating its radius of convergence. For the power series $\sum a_n x^n$, let $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = l$. Determine the radius of convergence of the series. Discuss the nature of the series for $l = 0$. (2+3+1)

8. Obtain the power series expansion of $(1-x)^{-2}$ by term-by-term differentiation justifying all the steps (4)

9. Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} x^n$ such that $\sum_{n=0}^{\infty} a_n b_n x^n$ also has R as the radius of convergence (3)

10. Show that the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges at one end of its interval of convergence while diverges at the other end (3)

11. If the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ represent the same function in $|x|<R$, then show that $a_n = b_n \quad n = 0,1,2 \ldots$ (5)

12. If $\sum_{n=0}^{\infty} a_n$ is convergent, verify that the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for $|x|<1$ Construct example of a power series which converges at both the ends of its interval of convergence.

(5)

IV Riemann integration

1. Show that is a function is monotone on $[a,b]$, it is integrable over $[a,b]$ (3)

2. If f is continuous on an interval $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$, then show that $F'(x) = f(x)$ for all $x \in [a,b]$ (3)

3. Defining $\log_e x = \int_1^x \frac{dt}{t}$ $(x>0)$ prove from the definition that

$$\frac{x}{1+x} < \log(1+x) < x \quad (x>0)$$

(3)

4 Prove or disprove If $|f|$ is Riemann integrable on $[a,b]$, then f is also Riemann integrable on $[a,b]$. (3)

5. Let $f : [a,b] \to R$ be bounded on $[a,b]$. Prove that f is Riemann integrable on $[a,b]$ if and only if for each $\epsilon > 0$, there exists a partition P on $[a,b]$ such that $U(P;f) - L(P;f) < \epsilon$. (5)

6 Let f $[a,b] \to R$ be a bounded function. If the set of points of discontinuity of f has a finite number of limit points in $[a,b]$ then prove that f is Riemann integrable on $[a,b]$. (4)

7. Let f $[a,b] \to R$ be integrable on $[a,b]$ and f possesses a primitive φ on $[a,b]$. Prove that

$$\int_a^b f(x)\,dx = \varphi(b) - \varphi(a).$$

(4)

8. Give examples of a function which is Riemann integrable, but has no primitive and also of a function which has primitive, but is not Riemann integrable (4)

9 Using second Mean Value theorem of integral calculus (Weierstrass form), show that

$$\left| \int_a^b \frac{\cos x}{1+x}\,dx \right| < \frac{4}{1+a} \quad \text{where } b > a > 0.$$

(3)

10. Let $f : [a,b] \to R$ be Riemann integrable on $[a,b]$ and the infimum of f is a positive real number. Prove that $\frac{1}{f}$ is Riemann integrable on $[a,b]$ (3)

11 Let f $[a,b] \to R$ be integrable over $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. If there exists a point $c \in (a,b)$ such that f is continuous at c and $f(c) > 0$, prove that $\int_a^b f(x)\,dx > 0$ (4)

12. Prove that $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\,dx < \frac{2\pi^2}{9}$ (3)

13. A function g is continuous on $[a,b]$ and $g(x) = \int_a^x g(t)\,dt$ Prove that $g(x)=0$ for all $x \in [a,b]$. (3)

14 Let f be defined on $[0,1]$ by
$f(0)=0$ and
$f(x) = (-1)^{r-1}, \frac{1}{r+1} < x \leq \frac{1}{r}$ for $r=1,2$
Is f integrable ? Justify your answer. (3)

15. A function f is defined on $[0,1]$ by
$f(x) = \sqrt{1-x^2}\,,\ x \in [0,1] \cap Q$
$= 1-x\,,\ x \in [0,1] - Q$
Show that f is not integrable over $[0,1]$ (4)

16. $\varphi(x) = \int_{x^2}^{x^3} \frac{1}{(1+t^2)^3}\,dt\,,\ x \in [1,\infty)$ Find $\varphi'(x)$ (3)

17 A function f is continuous on R and $\int_{-x}^{x} f(t)\,dt = 2\int_0^x f(t)\,dt$ for all $x \in R$ Prove that f is an even function on R (3)

18. Use first Mean Value theorem to prove that
$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \leq \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{k^2}{4}}}\,,\ k^2<1$ (4)

19. If e is defined by the equation $\int_1^e \frac{dt}{t} = 1$, prove that $2 < e < 3$ (3)

20. Let $f\ [a,b] \to R$, $g : [a,b] \to R$ be integrable on $[a,b]$. Prove that $\max(f,g)\ [a,b] \to R$ is integrable on $[a,b]$. (3)

21. If a function f is continuous on a closed interval $[a,b]$ and $\int_a^b fg\,dx = 0$ for every continuous function g on $[a,b]$, prove that $f(x)=0$ for all $x \in [a,b]$ (4)

22 A function f is continuous for all $x \geq 0$ and $f(x) \neq 0$ for all $x>0$ If $\{f(x)\}^2 = 2\int_0^x f(t)\,dt$ prove that $f(x)=x$ for all $x \geq 0$. (4)

23. Evaluate $\lim_{x\to 2} \frac{\int_2^x e^{\sqrt{1+t^2}}\,dt}{x-2} = e^{\sqrt{5}}$ (3)

24. For $x \geq 0$, let $\varphi(x) = \lim_{n\to\infty} \frac{x^n+2}{x^n+1}$ and $f(x) = \int_0^x \varphi(t)\,dt$ Show that f is continuous at 1 but not differentiable at 1

25. Evaluate $\int_0^4 ([x]+1)\,dx$ (3)

(3)

V Improper integral

1. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$ is convergent for $m>0$, $n>-1$ (4)

2. If f and g be positive functions on $[a,b]$ such that $\underset{x\to a+0}{Lt}\frac{f(x)}{g(x)} = l$, where l is a nonzero finite number, then prove that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or diverge together (5)

3. Show that the integral $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{x^n}\,dx$ exists if and only if $n< m+1$ (4)

4. Prove that the improper integral $\int_a^b f(x)\,dx$ converges at a if and only if to every $\epsilon>0$, there corresponds $\delta>0$ such that

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \epsilon \quad , \; 0<\lambda_1<\lambda_2<\delta$$ (4)

5. Prove that every absolutely convergent integral is convergent. Is the converse true? Justify your answer (4)

6. Test the convergence of the integral $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{\frac{1}{2}}}\,dx$. (3)

7 Show that $\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx$ converges if and only if $p>-1$, $q>-1$ (4)

8. Prove that the integral $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges for $p>0$, but absolutely converges for $p>1$ (3)

9 Test the convergence of $\int_0^\infty \frac{\sin x^m}{x^n}\,dx$ (3)

10. The function f is defined on $[0,\infty]$ by
$$f(x) = (-1)^{n-1}, \quad n-1\le x<n \quad , n\in\mathbb{N}$$
Show that the integral $\int_0^\infty f(x)\,dx$ does not converge (3)

11. Show that $\Gamma(x) > \frac{1}{e}\int_0^1 t^{x-1}\,dt$, $x>0$ and hence deduce that $\underset{x\to 0+}{Lt}\Gamma(x) = \infty$ (4)

12 Let φ be continuous on $[0,\infty)$ and $\underset{x\to 0+}{Lt}\varphi(x) = \varphi_0$, $\underset{x\to\infty}{Lt}\varphi(x) = \varphi_1$ Prove that $\int_0^\infty \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = (\varphi_0-\varphi_1)\ln\frac{b}{a}$, $b>a>0$ (4)

13. Prove that $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $m>0, n>0$ (4)

14 Show that $\int_{-1}^{1} \frac{dx}{x^3}$ is not convergent but its Cauchy principle value exists (3)

15 If $\int_{1}^{\infty} f(x)\,dx$ converges and $\lim_{x\to\infty} f(x) = L$, prove that $L = 0$.

(3)

16. Assume that f is Riemann integrable on $[a,b]$ and there is positive constant M such that $\int_a^b |f(x)|\,dx \leq M$ for every $b \geq a$ Show that $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} |f(x)|\,dx$ are convergent as improper integral. (4)

VI Functions of bounded variation

1 Let $f : [a,b] \to R$ be a function of bounded variation on $[a,b]$. Prove that f is bounded on $[a,b]$ Is the converse of the result true? Justify your answer. (4)

2. Prove that if $f\ [a,b] \to R$ be monotone on $[a,b]$, f is a function of bounded variation on $[a,b]$ (3)

3. Let $f : [a,b] \to R$ be continuous on $[a,b]$, f' exists and be bounded on (a,b). Prove that f is a function of bounded variation on $[a,b]$ (3)

4 Let $f : [a,b] \to R$ and $g : [a,b] \to R$ be functions of bounded variation on $[a,b]$ Prove that fg is a function of bounded variation on $[a,b]$ (3)

5. Give an example of a function f continuous on a closed interval $[a,b]$, but f is not a function of bounded variation on $[a,b]$ (3)

6. Give an example of a function f not continuous on a closed interval $[a,b]$, but f is a function of bounded variation on $[a,b]$ (3)

7. Let $f : [a,b] \to R$. Prove that f is a function of bounded variation on $[a,b]$ if and only if f can be expressed as the difference of two monotone increasing functions on $[a,b]$ (5)

8. $f(x) = x^2$, $x \in [-1,1]$ Show that f is a function of bounded variation on $[-1,1]$ Express f as the difference of two monotone increasing functions on $[-1,1]$ (3)

9. Let $f\ [a,b] \to R$ be a function of bounded variation on $[a,b]$. Prove that f can have only discontinuities of first kind. (3)

10. Let $f\ [a,b] \to R$ be a function of bounded variation on $[a,b]$ Define positive variation function $p(x)$ and negative variation function $n(x)$ Prove that

$$p(x) + n(x) = V(x) \text{ for all } x \in [a,b].$$

(5)

11. Let $f(x) = x - [x]$, $x \in [1,3]$ show that f is a function of bounded variation on $[1,3]$. Find the positive variation function, negative variation function (4)

12 Let $x_1, x_2, \ldots x_n$ be an enumeration of all rational numbers in $[0,1]$ and let $f : [0,1] \to R$ be defined by

$$f(x_n) = \frac{1}{n^2}, \quad n = 1, 2$$
$$= 0, \quad \text{elsewhere}$$

Prove that f is a function of bounded variation on $[0,1]$

(4)

13. Let $f(x) = \sin x + \cos x$, $x \in [0, \frac{\pi}{2}]$ Show that f is a function of bounded variation. Find the variation function V on $[0, \frac{\pi}{2}]$

(4)

VII Integral containing an arbitrary parameter

1. Use Weierstrass M-test to show that $\int_0^\infty e^{-xt}\frac{\sin t}{t}\,dt$ converges uniformly for all $x \geq 0$ (3)

2. Stating the reason for the validity of differentiation under the sign of integration prove that
$$\int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \pi\log\left(\frac{a+b}{2}\right) \quad a,b>0.$$
(5)

3. Establish the right to integrate under integral sign: $\int_0^\infty e^{-xy}\cos mx\,dx$, $y>0$ and deduce that
$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos mx\,dx = \frac{1}{2}\log\frac{b^2+m^2}{a^2+m^2}$$
(5) $a,b>0$

4. Establish the right to differentiate under the integral sign: $I = \int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx \quad a\in R$

and hence show that $I = \frac{\sqrt{\pi}}{2}e^{\mp 2|a|}$ (5)

5. Stating the reasons for the validity of differentiation under the sign of integration, prove that for $\alpha<1$,
$$\int_0^\pi \log(1+\alpha\cos x)\,dx = \pi\log\left(\frac{1+\sqrt{1-\alpha^2}}{2}\right)$$

6. Stating the reason for the validity of differentiation under the sign of integration prove that for $a>0$ (5)
$$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{1}{2}\ln(1+a^2)\tan^{-1}a$$
(5)

7. Show that $\lim_{\alpha\to 0}\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx = \frac{\pi}{2}$ (3)

8. Assuming $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$, $p>0$ show that
$$\int_0^\infty \frac{\cos ax-\cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a) \quad b>a>0.$$
(3)

9. Given that $\int_0^\infty \frac{dx}{1+\alpha^2x^2} = \frac{\pi}{2}\frac{1}{\alpha}$

show that
$$\int_0^\infty \frac{\tan^{-1}bx-\tan^{-1}ax}{x}\,dx = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$$
(3)

10. Show that $\int_0^\theta \log(1+\tan\theta\tan x)\,dx = \theta\log\sec\theta$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ (5)

VII Function of several variable

1. State a set of sufficient conditions for the validity of the following relation and establish it

$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{\underline{2}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$
$+ \frac{1}{\underline{3}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(a+\theta h, b+\theta k) \qquad 0<\theta<1$ (2+3)

2. Let $F(x,y) = \sin x \cos y$ Prove that there exists a $\theta \in (0,1)$ for which $\frac{3}{4} = \frac{\pi}{3} \cos\frac{\pi\theta}{3} - \frac{\pi\theta}{6} - \frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}$ holds (3)

3. Show that $f(x,y) = xy$ has a saddle point at $(0,0)$. (2)

4. Use Lagrange's method to find the minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3$ (4)

5. State and prove Mean Value theorem for a function of two independent variables. (4)

6. Use Lagranges method to find the shortest distance from the point $(0,b)$ to the parabola $x^2=4y$. (5)

7. Expand xy^2+3x-2 in Taylor's series in a neighbourhood of $(-1,2)$ (4)

8. Show that if $f(x,y) = 2x^4 - 3x^2y + y^2$, then $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$, but f has neither a maximum nor a minimum at $(0,0)$ (4)

9. Show that the points on the ellipse $5x^2-6xy+5y^2=4$ for which the tangent is at the greatest distance from the origin are $(1,1)$ and $(-1,-1)$. (6)

10. Find a point within a triangle such that the sum of the squares of its distances from the vertices is a minimum (5)

11 Using Taylor's theorem show that

$$\lim_{\substack{(x,y)\to(0,0)\\(y=-x)}} \frac{\sin xy + xe^x - y}{x\cos y + \sin 2y} = 2 \qquad (5)$$

12 Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2 = 225$ (4)

APPLICATION OF DIFFERENTIAL CALCULUS

I. Tangent, normal and related topics,

1. Find the points on the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ where the tangents are parallel to $2y + x = 0$. (5)

2. Show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to focus is $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ (5)

3. In the curve $x^m y^n = a^{m+n}$ prove that the portion of the tangent intercepted between the axes is divided at its point of contact into two segments which are in a constant ratio. (5)

4. Show that at any point on the curve $x^{m+n} = K^{m-n} y^{2n}$ the mth power of the subtangent varies as the nth power of the subnormal. (5)

5. Show that for the curve $by^2 = (x+a)^3$ the square of the subtangent varies as the subnormal. (5)

6. Find the condition that the curves $ax^3 + by^3 = 1$ and $cx^3 + dx^3 = 1$ should cut orthogonally (5)

7. Prove that the condition that $x \cos\alpha + y \sin\alpha = p$ should touch $x^m y^n = a^{m+n}$ is
$$p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha \cos^m\alpha$$ (5)

8. If $lx + my = 1$ is a normal to the parabola $y^2 = 4ax$, then show that $al^3 + 2alm^2 = m^3$. (5)

9. Find the pedal equation of the parabola $y^2 = 4ax$ wrt its vertex (5)

10. Define pedal equation of a curve. Find the pedal equation of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (5)

11. Find the pedal equation of a ~~our~~ circle with respect to a point on the circumference (5)

12. Find the pedal equation of the cardioide $r = a(1 + \cos\theta)$ (5)

13. Define the pedal of a curve Find the pedal of the cardioide $r = a(1+\cos\theta)$ (5)

14. Find the cartesian equation of the pedal of the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}} = 1$ with respect to the origin (5)

15. Find the points on the curve $xy^2 = (x+y)^2$ where the line $x = k$ cuts it orthogonally (5)

16. Show that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'(\theta - \frac{\pi}{2})$.

(5)

II Curvature and envelope

1 Prove that the centre of curvature of a point P of a curve is limiting position of the point of intersection of the normal to the curve at P with the normal to the curve at a neighbouring point Q on the curve as Q tends to P along the curve (5)

2. Show that for the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ the radius of curvature at any point is twice the portion of the normal intercepted between the curve and the axis of x (5)

3. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that, $\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}$. (5)

4. If ρ_1 and ρ_2 be the radii of curvature at the ends P and D of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then $\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} = (a^2+b^2)/(ab)^{\frac{2}{3}}$. (5)

5. Show that the chord of curvature parallel to the y axis for the curve $y = a \log \sec(x/a)$ is constant (5)

6. Show that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the radius of curvature at an extremity of the major axis is equal to half the latus rectum (3)

7. Find the radius of curvature at the origin, of the following curves

(i) $y^2 = x^2(a+x)/(a-x)$

(ii) $y^2 - 3xy - 4x^2 + 5x^3 + x^4 y - y^5 = 0$

(iii) $y^2 - 2xy - 3x^2 - 4x^3 - x^2 y^2 = 0$ (5)

8. Find the equation of circle of curvature of the curve $y = x^3 + 2x^2 + x + 1$ at $(0,1)$ (5)

9 Find the equation of circle of curvature of the curve $y = x^2 - 6x + 10$ at $(3,1)$ (5)

10. Find the evolute of the parabola $y^2 = 4ax$.

11 Find the evolute of the curve $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

12. Find the envelope of the curves $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ where the two parameters a and b are connected by the relation $a+b=c$, c being a constant. (5)

13 Given that $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ is the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$, find the necessary relation between a and b. (5)

14. Prove that the envelope of the normals to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is given by
$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$
(5)

15. Find the envelope of the family of straight lines drawn at right angles to the radii vectors of the cardiodie $r=a(1+\cos\theta)$ through their extremities (5)

16. From any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ perpendiculars PM, PN are drawn upon the coordinate axes. Show that MN always touches the curve
$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$$
(5)

17. If the centre of a circle lies upon the parabola $y^2=4ax$ and the circle passes through the vertex of the parabola, show that the envelope of the circle is
$$y^2(2a+x)+x^3=0$$
(5)

18. Show that the envelope of the ellipses
$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$$
where the parameters α, β are connected by the relation $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$. (5)

19. Find the envelope of the circles described on the radii vectors of the curve $r^m=a^m\cos m\theta$ as diameters. (5)

20. Show that the envelope of the circles
$$x^2+y^2-2\alpha x-2\beta y+\beta^2=0$$
where α, β are parameters and whose centres lie on the parabola $y^2=4ax$ is $x(x^2+y^2-2ax)=0$ (5)

III Asymptote

1. If a curve passes through the points (0,0), (1,0), (0,1) and if it has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$, find its equation. (5)

2. Show that the points of intersection of the curve $2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$ and its asymptotes lie on the ~~curve~~ line $8x + 2y + 1 = 0$. (5)

3. Find the asymptotes of the curve given by
$$y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$$
(5)

4. Find the asymptotes of the curve
$$(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$$
(5)

5. Find the asymptote (if any) of the curve
$$x = \frac{t^2}{1+t^3}, \quad y = \frac{t^2+2}{1+t}$$
where t is the parameter. (5)

6. Define rectilinear asymptote to a curve. Find the rectilinear asymptotes to the curve $y = xe^{\frac{1}{x^2}}$. (5)

7. If any of the asymptotes of the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ $(h^2 > ab)$ passes through the origin, prove that
$$af^2 + bg^2 = 2fgh.$$
(5)

8. Show that the asymptotes of the curve
$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$
form a square two if whose angular points lie on the curve. (5)

9. Show that the curve $y = \begin{cases} \sqrt{1+x^2}\,\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has no asymptote parallel to the y-axis and its only asymptotes are $y = \pm 1$.

(5)

IV Concavity, convexity, singular points.

1. Show that the origin is a node, cusp, or isolated point on the curve $y^2 = ax^2 + bx^3$ according as $a > 0$ or $=$ or < 0 (5)

2. Show that the points of inflexion on the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$. (5)

3. Show that the curve $y(x^2+a^2) = a^2x$ has three points of inflexion which lie on a straight line. (5)

4. Find the position and nature of the double points of the curve $y(y-6) = x^2(x-2)^3 - 9$. Find also, the equation of the tangent at the double point if the tangent is real. (5)

5. Prove that the curve $y = \log x$ is convex to the ~~ordin~~ foot of the ordinate in the range $0<x<1$ and convex when $x>1$ Prove also that the curve is convex with respect to any point on the y-axis. (5)

6. Prove that the curve $y^2 - 2x^2y - x^4y - x^4 = 0$ has a double cusp of first species at the origin. (5)

7. Find the points of inflexion of the curve $y = e^x(\sin x + \cos x)$ in the range $(0, 2\pi)$. (5)

8 Show that the curve $2y^2 = x^2y + x^3$ has a cusp of the first species at the origin and $x+y=2$ is an asymptote of it which cuts the curve at $(1,1)$ (5)

9. Show that the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards for $-2<x<0$ and $2<x<\infty$ but convex upwards for $-\infty<x<-2$ and $0<x<2$ Also find its points of inflexion (5)

10. Examine the curve given by $y = x^4 - 2x^2 + 1$ for concavity and convexity Also determine its points of inflexion (5)

11. Show that the curve $(x+y)^3 - \sqrt{2}(y-x+2)^2 = 0$ has a single cusp of first species at the point $(1,-1)$ (5)

INTEGRAL CALCULUS

I. Reduction formula, Beta and Gamma functions

1. Obtain the reduction formula for $\int \sin^m x \cos^n x \, dx$ where either m or n or both are negative integers (4)

2. If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx \quad (n>1)$, show that

$$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

(4)

3. Find a reduction formula for $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx \, dx$ Hence show that $I_{m,m} = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m}\right]$ (4)

4. Obtain a reduction formula for $\int \frac{dx}{(x^2+a^2)^n}$, n being a positive integer greater than 1. Hence evaluate $\int \frac{dx}{(x^2+a^2)^3}$ (4)

5. If $I_n = \int \frac{\sin nx}{\sin x} dx$, show that $(n-1)\left[I_n - I_{n-2}\right] = 2\sin(n-1)x$ (3)

6. Prove by integration by parts that if $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$ where m, n are positive integers, then $(m+n+1) I_{m,n} = n I_{m,n-1}$ and deduce that

$$I_{m,n} = \frac{\lfloor m \lfloor n}{\lfloor m+n+1}$$

(4)

7. If $I_n = \int_0^1 (1-x^2)^n dx$, prove that $(2n+1) I_n = 2n I_{n-1}$ Hence find I_n (4)

8. If $I_n = \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin nx \, dx$ show that

$$2(n-1) I_n = 1 + (n-2) I_{n-1}$$

and deduce that $I_n = \frac{1}{n-1}$ (4)

9. Define Beta function. Prove that

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx \quad (m,n>0)$$

(4)

10. Prove that $\Gamma(x+1) = x\Gamma(x) \quad (x>0)$ Hence show that $\Gamma(n+1) = \lfloor n$ (3)

11. Prove that $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$

What are the restriction on m, n? (4)

12. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$

$= (b-a)^{m+n-1} \; B(m,n) \qquad m,n>0.$

13. Establish the formula (4)

$\Gamma\left(\frac{1}{2}\right)\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right).$ (4)

14. Show that $\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$ (2)

15. Show that $\int_0^1 x^3 (1-x^2)^{\frac{5}{2}}\,dx = \frac{2}{63}$ (3)

16. When m, n and a be positive integers, show that

$$\int_0^{\infty} x^m e^{-ax^n}\,dx = \frac{1}{n a^{\frac{m+1}{n}}}\,\Gamma\left(\frac{m+1}{n}\right)$$ (3)

17 Assuming $\Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi \quad 0<m<1$, show that

$$\int_0^{\infty} x e^{-x^8}\,dx \times \int_0^{\infty} x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$$ (4)

18. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}\,dx = \frac{B(m,n)}{(b+c)^m b^n}$

$m, n, b, b+c > 0.$ (4)

II Area of a plane curve and related topics

1. Find the area of the region bounded by the curve $y = x(x-1)(x-2)$ and the x-axis (5)

2. Find the area in the first quadrant bounded by $x=0$, $y=0$ and $\sqrt{x}+\sqrt{y}=\sqrt{a}$. (5)

3. Show that the area bounded by $y=x^2$ and $x=y^2$ is $\frac{1}{3}$ (5)

4. Find the area of a loop of the curve
$$x(x^2+y^2) = a(x^2-y^2).$$ (5)

5. Find the area included between the curve $x^2y^2 = a^2(y^2-x^2)$ and its asymptotes (5)

6. Prove that the area included between the ~~funel~~ curve $x^3+y^3 = 3axy$ and its asymptotes $x+y+a=0$ is equal to the area of the loop. (5)

7. Prove that the area between the curve $y^2(a+x) = (a-x)^3$ and its asymptotes is $3a^2\pi$ square units (5)

8. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the line $y=a$ Using Pappus theorem show that the volume of the solid generated is $\frac{8\sqrt{2}\pi a^3}{15}$ (5)

9. A quadrant of a circle of radius a revolves about the tangent at one extremity show that the area of the curved surface generated is $\pi(\pi-2)a^2$, (5)

10. Show that the surface area of the solid generated by revolving the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ about the line $y=0$ is $\frac{64}{3}\pi a^2$, (5)

11. Find the centroid of a loop of the curve
$$y^2(a+x) = x^2(a-x)$$ (5)

12. Find the centroid of the area in the first quadrant bounded by $y=x^2$ and $y=x^3$. (5)

13. Find the centroid of the whole arc of the cardioide
$$r = a(1+\cos\theta)$$ (5)

14 Find the moment of inertia of the circumference of a circle of radius a about a diameter (5)

FUNCTION OF SEVERAL VARIABLE

1. When is a point (a,b) said to be a limiting point of a set A of $R \times R$? If $B = \{(a,0) ; a \in R\}$ show that B is a closed set but not an open subset of $R \times R$ (5)

2. If $S = \{(x,y) ; 0 < x < 1, 0 < y < 1\}$, show that S is an open set in $R \times R$. Mention one limit point of S which does not belong to S. (3)

3. Let $S = \{(x,y) ; 0 \leq x^2 + y^2 < 1\}$
$N = \{(x,y) ; -\frac{1}{2} < x, y < \frac{1}{2}\}$
Is N an open neighbourhood of $(0,0)$ lying in S? Justify your answer. (5)

4. Let $f(x,y) = \begin{cases} x \sin\frac{1}{y} + y \sin\frac{1}{x} & , xy \neq 0 \\ 0 & , xy = 0 \end{cases}$
Show that the limit exist at the origin but the repeated limit does not exist (5)

5. $f(x,y) = \begin{cases} x \sin\frac{1}{y} + \dfrac{x^2 - y^2}{x^2 + y^2} & , \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$
Show that $\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x,y)$ exists but $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ and $\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x,y)$ does not exist. (5)

6. Find $\lim\limits_{\substack{x \to 0 \\ y \to 0}} xy \dfrac{x^2 - y^2}{x^2 + y^2}$ using ϵ-δ definition. (5)

7. When is a function $f(x,y)$ said to be continuous at (a,b)?
If $f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y} & , x \neq y \\ = 0 & , x = y \end{cases}$
show that f is not continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ exist (5)

8. ~~If~~ Let f be a real valued function of two real variables and (a,b) is an interior point of domain of f. Assume that both the first order partial derivatives of f exist at (a,b) and one of them is bounded in some neighbourhood of (a,b). Prove that f is continuous at (a,b) (5)

9. Let f be continuous function in $[a-\delta_1, a+\delta_1]$ and g be continuous in $[b-\delta_2, b+\delta_2]$. Let $h(x,y) = \min\{f(x), g(y)\}$

Show that h is continuous in the region N where $N = \{(x,y) \in \mathbb{R}^2 ; a-\delta_1 \le x \le a+\delta_1 , b-\delta_2 \le y \le b+\delta_2\}$. (5)

10. Prove that, if a function f is differentiable at an interior point of this domain, first order partial derivatives exist at that point and f is continuous at that point. (5)

11. Let $f(x,y) = (|x+y| + x+y)^k \quad \forall (x,y) \in \mathbb{R}^2$. Determine the values of k for which f is differentiable everywhere (5)

12. Show that the function $|x|+|y|$ is continuous, but not differentiable at the origin. (5)

13. Let
$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{if } xy \neq 0 \\ x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin\frac{1}{y} & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

Show that f is differentiable at $(0,0)$, but f_x, f_y are not continuous at $(0,0)$ (5)

14. Let
$$f(x,y) = \frac{x^3 - y^3}{x^2+y^2} \qquad (x,y) \neq (0,0)$$
$$= 0 \qquad (x,y) = (0,0)$$

Show that f is continuous at $(0,0)$, but is not differentiable at $(0,0)$ (5)

15. Prove that $f_{xy} \neq f_{yx}$ at origin for the function
$$f(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y) \quad (x \neq 0, y \neq 0)$$
$$= 0 \quad , \text{elsewhere}$$
(5)

16. Show that $z = x\cos(y/x) + \tan(y/x)$ satisfies $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = 0$, except at points for which $x = 0$. (5)

17. State and prove schwarz's theorem. (5)

18. Show that for the function
$$f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$f_{xy}(0,0) = f_{yx}(0,0)$ even though the conditions of schwarz's theorem ~~are~~ is not satisfied. (5)

19. Show that $f(xy, z-2x)=0$ satisfies under certain conditions the equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$ What are these conditions? (5)

20. Prove that by the transformation $u = x - ct$, $v = x + ct$ the partial differential equation $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ reduces to $\frac{\partial^2 z}{\partial u \partial v} = 0$ (5)

21. If z is a function of u and v and $u = x^2 - y^2 - 2xy$, $v = y$, prove that the equation $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ is equivalent to $\frac{\partial z}{\partial v} = 0$. (5)

22. Given that f is a function of x and y and that $x = u^2 v$, $y = uv^2$ prove that

$$2x^2 f_{xx} + 2y^2 f_{yy} + 5xy f_{xy} = uv f_{uv} - \frac{2}{3}(u f_u + v f_v).$$ (5)

23. State and prove converse of Euler's theorem for the function of three variables (5)

24. Let $\underset{(x,y)\to(a,b)}{Lt} f(x,y) = l$ and let $y = \varphi(x)$ be any function such that $\varphi(x) \to b$ when $x \to a$. Show that $\underset{x\to a}{Lt} f(x, \varphi(x))$ exists and is equal to l (3)

25. Let f be a homogeneous function of degree n having continuous second order partial derivatives. Prove that each of f_x, f_y, f_z are homogeneous function of degree $n-1$ (5)

26. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$$ (5)

27. Prove, using the method of jacobian, that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$ (5)

28. Let $f(x,y) = \begin{cases} xy & \text{if } |x| \geq |y| \\ -xy & \text{if } |x| < |y| \end{cases}$

Is $f_{xy} = f_{yx}$ at the origin?

29. If α, β, γ are the roots of the equation in t such that $\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1$

then prove that $\frac{\partial(u,v,w)}{\partial(\alpha,\beta,\gamma)} = -\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(b-c)(c-a)(a-b)}$

(5)

30. If $u = \frac{x^2+y^2+z^2}{x}$, $v = \frac{x^2+y^2+z^2}{y}$ and $w = \frac{x^2+y^2+z^2}{z}$

find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

(5)

31. If $x+y+z = u$, $y+z = uv$, $z = uvw$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

(5)

32. Show that $xy \sin x + \cos y = 0$ determines y uniquely as a function of x near $(0, \frac{\pi}{2})$ and find $\frac{dy}{dx}$ at $(0, \frac{\pi}{2})$.

(5)

33. Show that $2xy - \log xy = 2$ determines y uniquely as a function of x near $(1,1)$ and find $\frac{dy}{dx}$ at $(1,1)$

(5)

34. If u, v are functions of ξ, η, ζ and the variables ξ, η, ζ are functions of the independent variables x, y, then show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\eta,\zeta)}\frac{\partial(\eta,\zeta)}{\partial(x,y)} + \frac{\partial(u,v)}{(\zeta,\xi)}\frac{\partial(\zeta,\xi)}{\partial(x,y)}$$

(5)

35. Let u_1, u_2, u_3 be three real valued functions of three variables x_1, x_2, x_3. Assume that all the first order partial derivatives are continuous. If there exists a functional relation between u_1, u_2, u_3 prove that

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = 0$$

(5)

HYDROSTATICS

I. Specific gravity, density, pressure of heavy fluids

1. Explain the terms 'perfect fluid' and 'pressure at a point in the fluid'. (2)

2. Show that the pressure at a point in a fluid in equilibrium is the same in every direction. Does the result hold in a moving fluid? (7)

3. Prove that the specific gravity of a mixture of n liquids are greater when equal volumes are taken than when equal weights are taken, assuming that there is no change in volume due to mixing (7)

4. Prove that the free surface of a homogeneous liquid at rest under gravity is horizontal. Also show that the surface of separation of two liquids of different densities, which do not mix, at rest under gravity, is a horizontal plane (7)

5. A circular tube centred O is filled with three fluids of densities ρ_1, ρ_2, ρ_3 (in descending order of magnitude) and placed in a vertical plane If 2α, 2β, 2γ be the angles subtended at the centre by the fluids and P be the point on the circumference midway between the ends of the lightest fluid, then angle θ which OP makes with the vertical is given by

$$\frac{\rho_2-\rho_3}{\rho_1-\rho_3} = \frac{\sin\alpha}{\sin(\alpha+\theta)} \quad \frac{\sin(\beta-\theta)}{\sin\beta}$$ (8)

6 A fine parabolic tube is held with its axis vertical and vertex downwards. It contains liquids of successive densities ρ_1, ρ_2 ρ_n beginning from the top on one side and ending with the last liquid on the other. If $r_0, r_1,$ r_n be the focal distances of the boundaries, prove that

$$\rho_1(r_0-r_1)+\rho_2(r_1-r_2)+ \quad +\rho_n(r_{n-1}-r_n)=0.$$ (8)

7 A fine uniform cycloidal tube is held with its axis vertical and vertex downwards and equal weights of two liquids occupy lengths a and b. Prove that the heights of the free surfaces of the liquids above the vertex are in the ratio $(3a+b)^2$ $(3b+a)^2$. (8)

II Equilibrium of fluids in given field of forces

1. Prove that the necessary and sufficient condition for equilibrium of a fluid under the action of external force $\vec{F}$ is $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ (8)

2. Prove that in a conservative field of force, the surfaces of equipressure, equidensity and equipotential coincides Hence show that for a fluid in equilibrium under the action of gravity, these surfaces are horizontal (8)

3. A given volume V of liquid is acted upon by forces $-\frac{\mu x}{a^2}$, $-\frac{\mu y}{b^2}$, $-\frac{\mu z}{c^2}$ Find the equation to the free surface. (7)

4. If a given volume of fluid is at rest under forces whose components per unit mass are $\lambda y(a-z)$, $\lambda x(a-z)$; μxy, show that the density must be proportional to $\frac{1}{xy(a-z)}$ (7)

5. A spherical vessel is just filled with a heavy liquid, the particle of which attract one another according to the law of gravitation. If the pressure at the highest point vanishes, show that the resultant thrust across a vertical diametral plane is
$$\pi g\rho a^3 + \frac{1}{3}\pi^2\rho^2\gamma a^4$$
where a is the radius, ρ the density and γ is constant of gravitation (8)

6. A mass of fluid rests upon a plane subject to a central attractive force $\frac{\mu}{r^2}$ situated at a distance c from the plane on the side opposite to that on which the fluid is, a is the radius of the free surface of the fluid. Show that the pressure on the plane is
$$\frac{\pi\rho\mu(a-c)^2}{a}$$
(8)

7. If the components parallel to the axis of forces acting on the element of fluid at (x,y,z) be proportional to $y^2+2\lambda yz+z^2$, $z^2+2\eta zx+x^2$, $x^2+2\mu xy+y^2$, then show that if equilibrium is possible, we must have $2\lambda = 2\mu = 2\nu = 1$ (7)

8. Prove that the surfaces of equipressure for a fluid rotating uniformly about an axis is a paraboloid of revolution What will happen in case of a heteogeneous fluid ? (7)

9 A hollow circular cylinder of radius r, height h and effective specific gravity σ, floats in water which is rotating round a vertical axis coincident with the axis of the cylinder; and the cylinder is open at its upper end Show that the maximum angular velocity ω of the fluid so that the cylinder shall not sink is given by $r^2\omega^2 = 4gh(1-\sigma)$ (8)

10 Fluid is contained within a circular tube of radius a in a vertical plane which can rotate about a vertical axis If the fluid subtends an angle θ at the centre, the least angular velocity to make the fluid divide into two parts is $\sqrt{\frac{g}{a}}\sec\frac{\theta}{4}$ (7)

11. A hollow sphere of radius a half filled with liquid, is made to rotate with angular velocity ω about its vertical diameter. If the lowest point of the sphere is just exposed, show that $2g = a\omega^2(2-\sqrt[3]{4})$ (7)

12. A circular cylinder of radius a, whose axis is vertical is filled to a depth h with homogeneous liquid of density g. A piston of weight $\pi g g \alpha a^2$ which works in the cylinder without friction is placed on the top of the fluid. Show that if the cylinder and liquid rotate about its axis with gradually increasing velocity ω, the piston will begin to rise when $\omega a = 2\sqrt{g\alpha}$ (8).

13 A tube of small section is in the form of three sides of a square of side a, of which the middle one is horizontal It is filled with water and revolves with an angular velocity ω about a vertical side, prove that the amount which will flow out would fill a length $\frac{a^2\omega^2}{2g}$ or $a+\sqrt{a^2-\frac{2ag}{\omega^2}}$ according as $\omega <$ or $> \sqrt{\frac{2g}{a}}$

(8)

III. Thrust on plane surface, Centre of pressure

1. A triangle ABC is immersed in a heavy homogeneous liquid at rest with the vertex A in the free surface and the median through A vertical. Prove that the ratio of the thrusts on the two parts into which the triangle is divided by the median is $\dfrac{5b^2+3c^2-2a^2}{3b^2+5c^2-2a^2}$ (7)

2. A rectangular area is immersed in a heavy liquid with two sides horizontal and is divided by horizontal lines into strips on which the total thrusts are equal. Prove that if p, q, r are the breadths of three consecutive strips $p(p+q)(q-r) = r(q+r)(p-q)$ (7)

3. A vertical circular cylinder of height $2h$ and radius r, closed at the top, is just filled by equal volumes of two liquids of densities ρ and σ $(\sigma > \rho)$. Show that if the axis be gradually inclined to the vertical, the pressure at the lowest point of the base will never exceed $g(\rho+\sigma)(r^2+h^2)^{\frac{1}{2}}$ (7)

4. What is the 'centre of pressure' for a surface immersed in a liquid? Prove with usual notation that the depth of the centre of pressure of a plane area below the horizontal line through the centroid of the area is $\dfrac{k^2}{\bar{h}}$ (8)

5. Find the depth of centre of pressure of a triangular area immersed in a homogeneous liquid with its plane vertical in terms of the depth of its vertices (7)

6. Prove that the position of centre of pressure does not change as the plane area is rotated about the line of intersection of its plane with the free surface. (7)

7. The side AB of a triangle ABC is in the surface of a fluid and points D, E are taken on AC such that the pressure on the triangles BAD, BDE, BEC are equal. Show that $AD : DE : EC = 1 : (\sqrt{2}-1) : (\sqrt{3}-\sqrt{2})$. (7)

8. When the depth of the liquid is increased by an amount a, the depth of the C.P. is found to be increased by y, and when the depth of the liquid is increased by b, that of the C.P. is found to be increased by z. Show that the depth of C.G. in the original state of the liquid is $ab(b-a+y-z)/(az-by)$. (8)

9. A plane quadrilateral ABCD is entirely immersed in water with the side AB in the surface. If the depth of C and D below the surface are γ and δ respectively, and that of the C.G is h, prove that the depth of C.P. is

$$\frac{\gamma+\delta}{2} - \frac{\delta\gamma}{6h},$$

(8)

10. A quadrant of a circle is immersed in a liquid with ~~a~~ a bounding radius in the surface. Find the position of the centre of pressure

(7)

11. A square lamina is wholly immersed in a heavy homogeneous fluid with its plane vertical and one corner in the surface; if it be turned in its own plane about this corner and is always immersed, show that the locus of centre of pressure in the lamina is a straight line.

(8)

12. Show that if a lamina totally immersed in a fluid is a quadrant of a circle of radius a, of which the ~~circ~~ centre is in the surface, the locus of the centre of pressure in the lamina lies on a straight line of length $3a\sqrt{2}(\pi-2)/16$

(8)

13 One end of a horizontal pipe of circular section is closed by a vertical circular door hinged to the pipe at the top. Find the moment about the hinge of the liquid thrust on the door when the pipe is half full of liquid

(8)

14 An ellipse of eccentricity $\frac{1}{4}$ is just immersed with its major axis vertical. Show that the centre of pressure coincides with a focus

(7)

15. A parallelogram is immersed vertically in a liquid with a corner in the surface. If a and b be the depths of the adjacent corners, prove that the depth of the centre of pressure is $\dfrac{2a^2+3ab+2b^2}{3(a+b)}$

(8)

IV Thrust on curved surfaces

1 How to determine the thrust due to the liquid on a curved surface wholly immersed in a heavy homogeneous liquid at rest (7)

2. Define superincumbent fluid. Find the resultant thrust on a curved surface bounded by a plane curve wholly immersed in a heavy homogeneous liquid at rest (7)

3. A cone whose vectorial angle is 2α, has its lowest generator horizontal and is filled with liquid; prove that the resultant thrust on the curved surface is $\sqrt{1+15\sin^2\alpha}$ times the weight of the liquid. (8)

4. A solid right circular cone of vertical angle $60°$ is just immersed in water so that one generator is in the surface of the liquid. Prove that the resultant thrust on the curved surface of the cone is to the weight of the water displaced by the cone is $\sqrt{7}:2$ (8)

5. A hollow cone without weight, closed and filled with some liquid, is suspended from a point in the rim of its base. If θ be the angle which the direction of the resultant pressure makes with the vertical then show that

$$\cot\theta = \frac{28\cot\alpha + \cot^3\alpha}{48}$$ (8)

6. A solid right circular cone is divided into two parts by a plane through its axis and one of these portions is just immersed, vertex downwards in water. Find the resultant thrust on its curved surface, and show that it is inclined at an angle $\tan^{-1}\left(\frac{\pi}{2}\tan\alpha\right)$ to the horizontal, where α is the semivertical angle of the cone. (8)

7. A spherical shell formed of two halves in contact along a vertical plane is filled with water. Show that the resultant pressure on either half of the shell is $\frac{\sqrt{13}}{4}$ of the total weight of the liquid (7)

V Equilibrium and stability of floating bodies

1, State the conditions of equilibrium of a body freely floating in a liquid. (2)

2. Define metacentre With usual notation prove that $HM = \frac{AK^2}{V}$ (8)

3. A cone of given weight and volume floats in a given liquid with its vertex downwards Show that the surfaces of the cone in contact with the liquid is least when the vectorial angle of the cone is $2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (8)

4. A uniform prism whose cross section is an isosceles triangle of vertical angle 2α, floats freely in a liquid with its base just immersed, one edge being in the surface Show that the ratio of its density to that of the liquid is $2\sin^2\alpha$ (8)

5. A semicircular lamina has one of the end of its diameter smoothly hinged to a fixed point above the surface of a liquid, and floats with its plane vertical and its diameter half immersed. If the inclination of the diameter to the horizon is $\frac{\pi}{4}$, prove that the ratio of the density of the liquid to that of the lamina is $4(3\pi-4)$ $(9\pi-8)$ (8)

6. A rectangle movable about an angular point which is fixed below the surface of liquid, floats with its sides equally inclined to the vertical and with half its area immersed in the liquid show that the ratio of the density of the body to that of the liquid is $(a-b)$ $4a$ $(a>b)$ (7)

7. A rod of small section and of density ρ, has a small portion of metal of weight $\frac{1}{n}$ th that of the rod attached to one extremity Prove that the rod will float at any inclination in a liquid of density σ if $(n+1)^2\rho = n^2\sigma$. (7)

8. If a floating solid be a cylinder, with its axis vertical, the ratio of whose specific gravity to that of the fluid is σ, prove that the equilibrium will be stable, if the radius of the base to the height be greater than $[2\sigma(1-\sigma)]^{\frac{1}{2}}$ (8)

VI Gases

1. Explain the concept of convective equilibrium in atmosphere Show under such a condition the temperature decreases upwards uniformly at a constant rate α given by $\alpha = \frac{\gamma - 1}{\gamma} \frac{g}{R}$ (7)

2. Find the pressure in an isothermal atmosphere at a height z (i) when g is constant (ii) when g is variable (8)

3. If T be the absolute temperature at height z and T_0 be its value at sea level in the atmosphere having convective equilibrium of temperature and satisfying the Law $p = k\rho^{\gamma}$, prove that

$$\frac{T}{T_0} = 1 - \frac{\gamma - 1}{\gamma} \frac{rz}{H(r+z)}$$

where r is the radius of the earth, H the height of the homogeneous atmosphere (8)

4. What is an 'adiabatic change of state', Derive the relation pv^{γ} = constant for adiabatic expansion of a compressible fluid where the symbols have usual meaning (6)

5. A hollow gas-tight sphere containing hydrogen requires a force mg to prevent it from rising when the lowest point touches the ground, the total mass of the sphere and hydrogen is M Show that the sphere can float in equilibrium with its lowest point at a height h above the ground where $h = \frac{k}{g} \log \frac{M+m}{M}$, k being the ratio of the pressure of the atmosphere to its density. (8)

6. If near the earth's surface gravity be assumed to be constant, and the temperature in the atmosphere to be given by $t = t_0 \left(1 - \frac{z}{nH}\right)$ where H is the height of the homogeneous atmosphere, show that the pressure in the atmosphere will be given by the equation

$$p = p_0 \left(1 - \frac{z}{nH}\right)^n$$

(7)

9. Prove that a circular cylinder of radius a and length $\frac{a}{h}$ cannot float upright in stable equilibrium if its sp. gr. lies between

$$\frac{1}{2}[1-\sqrt{1-2h^2}] \text{ and } \frac{1}{2}[1+\sqrt{1-2h^2}] \qquad (8)$$

10. A cylinder floating with its axis horizontal and in the surface, is displaced in the vertical plane through the axis Discuss its stability of equilibrium. (7)

11. Show that a homogeneous right circular cone of vertical angle $60°$ cannot float stably with its axis vertical and vertex-downwards unless its density as compared with that of the liquid is greater than $\frac{27}{64}$ (7)

12. A hollow cone whose weight is half that of the water it can contain, floats in water in stable equilibrium with its axis vertical and vertex downwards. Prove that if α be the semivertical angle of the cone

$$\cos^6\alpha < \frac{729}{1024} \qquad (7)$$

13. A solid paraboloid of sp. gr. σ floats in a liquid of sp. gr. ρ with its axis vertical and vertex downwards. Show that the equilibrium will be stable if

$$3a > h\left(1-\sqrt{\frac{\sigma}{\rho}}\right)$$

where $4a$ is the latus rectum of the generating parabola and h is the height of the solid (8)

14. Show that, when a uniform hemisphere of density ρ and radius a floats with its plane base immersed in a homogeneous liquid of density σ, the equilibrium is stable and the metacentric height is

$$\frac{3}{8}a\,\frac{(\sigma-\rho)}{\rho} \qquad (8)$$

15. A cone of density σ, whose height is h and the radius of whose base is a floating with its axis vertical and vertex upwards in liquid of density ρ. Prove that the equilibrium is stable if

$$\frac{\sigma}{\rho} < 1-\cos^6\alpha \qquad (8)$$